Proof of Dirichlet's Theorem for Primes in Arithmetic Progression

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1 Euclid and Euler's Proof of the Infinitude of Primes

Around 300 B.C.E., Euclid proved one of the first profound results in number theory, namely that there are infinitely many prime numbers. His proof is known for its simplicity and can be stated rather quickly:

If there were only finitely many primes, then we can list them all out, multiply them all together, and add 1 to the product. This new number is not divisible by any of the primes in our list. However, by the fundamental theorem of arithmetic, it must be divisible by some prime number which is not in our list. Since our list contains all the prime numbers, we have a contradiction. Thus, there are infinitely many primes numbers.

Some 2000 years later, Euler found his own proof for the infinitude of the primes. In fact, his result was a little bit stronger. He had shown that the sums of the reciprocals of all the prime numbers diverges. This would then imply the infinitude of the primes, as the sum would be finite otherwise. His proof is as follows:

Consider the zeta function, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for s a real number greater than 1. This sum converges absolutely for $s > 1$, and moreover, Euler showed that this sum can be written as the following product over primes:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

For brevity, we will assume that any sums or products with index p is taken over primes. We can then take logarithms on both sides to obtain

$$\log \zeta(s) = -\sum_{p} \log \left(1 - \frac{1}{p^s}\right) = \sum_{p} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}},$$

where the last equality is obtained by using the power series representation of $\log(1-x)$. We can then separate the sum into two parts:

$$\log \zeta(s) = \sum_{p} \frac{1}{p^s} + \sum_{p} \sum_{k \geq 2} \frac{1}{kp^{ks}}.$$

Looking at the second sum at $s = 1$, we have that

$$\sum_{p} \sum_{k \geq 2} \frac{1}{kp^k} \leq \sum_{p} \sum_{k \geq 2} \frac{1}{p^k} = \sum_{p} \frac{1}{p(p-1)} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

We can see that the sum $\sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}}$ remains bounded as $s \to 1^+$ and we know that $\zeta(s)$ diverges as $s \to 1^+$. Thus, we have that

$$\sum_{p} \frac{1}{p} = \lim_{s \to 1^+} \sum_{p} \frac{1}{p^s} = \lim_{s \to 1^+} \log \zeta(s) - \lim_{s \to 1^+} \sum_{p} \sum_{k \geq 2} \frac{1}{kp^{ks}}.$$

We have shown that the last term is finite for $s \to 1^+$. Furthermore, we have that $\zeta(s) \to \infty$ for $s \to 1^+$, as this reduces to the harmonic series. Thus, $\sum_p \frac{1}{p}$ must diverge, which shows that there are infinitely many primes.

Euler's proof is a monumental first step towards Dirichlet's theorem. The main goal is essentially the same as Euler's, which is to show that the sums of the reciprocals of all primes in a certain arithmetic sequence diverge. To do this, we instead consider a generalization of $\zeta(s)$.

2 Dirichlet's Theorem for The Progressions $4n+1$ and $4n+3$

As a motivation for the proof of Dirichlet's theorem, we will look at a proof for the special cases of the primes which are 1 more and 1 less than a multiple of 4. The statement we will prove is that there are infinitely many primes in the arithmetic sequence $4n+1$ and $4n+3$. To do so, consider the function defined by

$$\chi(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{if } n \text{ is even} \end{cases}$$

It can be seen that the function χ is completely multiplicative, so that $\chi(mn) = \chi(m)\chi(n)$ for any two integers m and n. Next, we will consider the following function:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which is called the Dirichlet L-function for our function χ and serves the same purpose as the zeta function in Euler's proof. Since χ is multiplicative, $L(\chi, s)$ can be expressed as an infinite product over prime numbers, also called the Euler product:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

Let $f_1(s) = \sum_{p \equiv 1 \pmod 4} \frac{1}{p^s}$ and $f_2(s) = \sum_{p \equiv 3 \pmod 4} \frac{1}{p^s}$. Then, we have that $f_1(s) + f_2(s) = \sum_{p>2} \frac{1}{p^s}$ and $f_1(s) - f_2(s) = \sum_p \frac{\chi(p)}{p^s}$. In section 1, we saw how Euler showed that $\sum_p \frac{1}{p}$ diverges, so $f_1(s) + f_2(s)$ also diverges for $s \to 1^+$. If we can show that $f_1(s) - f_2(s)$ is bounded for $s \to 1^+$, then it would follow that both $f_1(s)$ and $f_2(s)$ diverge for $s \to 1^+$. From the power series for $\arctan x$, we have that

$$L(\chi, 1) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Taking the logarithm of the Euler product of $L(\chi, s)$, we have

$$\log L(\chi, s) = \sum_p \frac{\chi(p)}{p^s} + \sum_p \sum_{k \geq 2} \frac{\chi(p)^k}{kp^{ks}}.$$

The first term converges absolutely for $s > 1$ by comparison with the zeta function. At $s = 1$, the second sum is absolutely convergent:

$$\left| \sum_p \sum_{k \geq 2} \frac{\chi(p)^k}{kp^k} \right| \leq \sum_p \sum_{k \geq 2} \frac{|\chi(p)|^k}{kp^k} = \sum_{p>2} \sum_{k \geq 2} \frac{1}{kp^k} \leq \sum_p \sum_{k \geq 2} \frac{1}{kp^k} \leq 1,$$

where the last inequality was proved in section 1. Since $L(\chi, s)$ is continuous at $s = 1$ with $L(\chi, 1) = \frac{\pi}{4}$, we have that

$$\sum_p \frac{\chi(p)}{p} = \lim_{s \to 1^+} \sum_p \frac{\chi(p)}{p^s} = \lim_{s \to 1^+} \log L(\chi, s) - \lim_{s \to 1^+} \sum_p \sum_{k \geq 2} \frac{\chi(p)^k}{kp^{ks}}.$$

The limit on the right hand side exists since we have shown the sum on the right converges at $s = 1$ and $\lim_{s \to 1^+} \log L(\chi, s) = \log \frac{\pi}{4}$. Thus, $f_1(s) - f_2(s)$ is bounded for $s \to 1^+$, which proves that $\sum_{p \equiv 1 \pmod 4} \frac{1}{p}$ and $\sum_{p \equiv 3 \pmod 4} \frac{1}{p}$ both diverge. Thus, there are infinitely many prime numbers in the arithmetic sequence of numbers of the form $4n + 1$ and $4n + 3$.

Let's look at the main steps that were needed to prove this result:

- Firstly, we needed to find a multiplicative function $\chi(n)$ that helped filter out primes of the form $4n + 1$ and $4n + 3$.

- Then, we defined two functions $f_1(s) = \sum_{p\equiv 1 \pmod 4} \frac{1}{p^s}$ and $f_2(s) = \sum_{p\equiv 3 \pmod 4} \frac{1}{p^s}$, whose behaviors we wanted to analyze for $s \to 1^+$.
- We then considered a linear combination of f_1 and f_2, which we knew diverged at $s = 1$ and a linear combination of f_1 and f_2, whose behavior at $s = 1$, would imply the divergence of f_1 and f_2 at $s = 1$.
- Using the Euler product of $L(\chi, s)$ and the fact that $\log L(\chi, s)$ was well-behaved at $s = 1$, we were able to show that f_1 and f_2 diverge at $s = 1$.

This is practically the exact steps we will take to prove the general theorem. The functions $\chi(n)$ that we need are called the Dirichlet characters, and they are complex-valued. The linear combinations required in step 3 will require some Fourier analysis. Lastly, to analyze the behavior of $\log L(\chi, s)$ at $s = 1$ for some arbitrary character χ, we will need some complex analysis. The need for complex analysis to prove a result in number theory, is why Dirichlet's theorem is considered one of the first theorems in analytic number theory.

3 The Dirichlet Characters

The Dirichlet characters χ are homomorphisms from $(\mathbb{Z}/m\mathbb{Z})^\times$ to S^1, where S^1 is the multiplicative group of complex numbers with absolute value 1. Because χ is a homomorphism, we have that $\chi(ab) = \chi(a)\chi(b)$ for $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$. Furthermore,

$$\chi(a)^{\varphi(m)} = \chi\left(a^{\varphi(m)}\right) = \chi(1) = 1.$$

Thus, $\chi(a)$ is a root of unity. The Dirichlet character defined by $\chi(n) = 1$ for all $n \in (\mathbb{Z}/m\mathbb{Z})^\times$ is called the trivial character and is denoted $\chi_0(n)$. Any other character, which isn't identically 1, is called a nontrivial Dirichlet character (i.e., if there exists $n \in (\mathbb{Z}/m\mathbb{Z})^\times$ such that $\chi(n) \neq 1$, then χ is nontrivial).

We can generalize the concept of a Dirichlet character to any finite abelian group G by defining a character to be a homomorphism from G to S^1. A Dirichlet character is a character for the finite abelian group $(\mathbb{Z}/m\mathbb{Z})^\times$, and we will sometimes just write "χ is a Dirichlet character mod m" to refer to a Dirichlet character for $(\mathbb{Z}/m\mathbb{Z})^\times$. Using Lagrange's theorem, it can be shown that for any $g \in G$, $\chi(g)$ is a root of unity. For any finite abelian group G, we can also consider the set of characters acting on the elements of G. This set is called the dual of G, denoted $\hat{G}$, and it is also a finite abelian group. Here is why:

1. The product of two characters is another character:
   Let χ_1 and χ_2 be two characters in $\hat{G}$. Then for any $g_1, g_2 \in G$, we have that

   $$(\chi_1\chi_2)(g_1g_2) = \chi_1(g_1g_2)\chi_2(g_1g_2) = \chi_1(g_1)\chi_1(g_2)\chi_2(g_1)\chi_2(g_2).$$

Rearranging gives us

$$(\chi_1\chi_2)(g_1g_2) = \chi_1(g_1)\chi_2(g_1)\chi_1(g_2)\chi_2(g_2) = (\chi_1\chi_2)(g_1) \cdot (\chi_1\chi_2)(g_2),$$

which shows that $\chi_1\chi_2$ is a homomorphism, and thus, a character.

2. The identity in $\hat{G}$ is simply the trivial character χ_0:
   For any $g \in G$,
   $$\chi(g)\chi_0(g) = \chi(g),$$
   since the trivial character is identically 1 for all $g \in G$.

3. The inverse character of χ is its complex conjugate $\bar{\chi}$:
   For any $g \in G$, we have
   $$1 = \chi(gg^{-1}) = \chi(g)\chi(g)^{-1}$$
   Multiplying both sides by $\bar{\chi}$ gives
   $$\bar{\chi}(g) = |\chi(g)|^2\chi(g)^{-1} = \chi(g)^{-1},$$
   since $\chi(g) \in S^1$, and so it has absolute value 1. Moreover, $\bar{\chi}$ is also a character since we can write it in terms of the character χ, namely $\bar{\chi}(g) = \chi(g^{-1})$. Another way to show that $\bar{\chi}$ is a character is to use the fact that complex conjugation is multiplicative, so $\bar{\chi}$ is still a homomorphism.

Complex multiplication is associative and commutative, so the characters themselves form a group. Moreover, we have that $|\hat{G}| = |G|$, but this is a result we will take as granted. The following is an important lemma about characters:

Lemma 0.1. *Let G be a finite abelian group and χ a character for G. Then,*

$$\sum_{g\in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0 \end{cases},$$

$$\sum_{\chi\in \hat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1 \end{cases}.$$

Proof. If $\chi = \chi_0$ is trivial, then $\chi(g) = 1$ for all $g \in G$. Thus, $\sum_{g\in G} \chi(g) = \sum_{g\in G} 1 = |G|$. If $\chi \neq \chi_0$, then there exist $g_0 \in G$ such that $\chi(g_0) \neq 1$. Furthermore, the function $f : G \to G$ given by $f(g) = g_0g$ is a bijection, and thus, multiplying elements of G by another element g_0 simply permutes the elements. Thus,

$$\sum_{g\in G} \chi(g) = \sum_{g\in G} \chi(g_0g) = \chi(g_0) \sum_{g\in G} \chi(g).$$

Thus, $(1 - \chi(g_0)) \sum_{g\in G} \chi(g) = 0$. Since $\chi(g_0) \neq 1$, we have that $\sum_{g\in G} \chi(g) = 0$. The proof of the second sum in the lemma is similar. If $g = 1$, then $\chi(g) = 1$

for all $\chi \in G$ since χ is a homomorphism. Thus, $\sum_{\chi \in \hat{G}} \chi(g) = \sum_{\chi \in \hat{G}} 1 = |\hat{G}| = |G|$. If $g \neq 1$, then we can find another character ψ such that $\psi(g) \neq 1$. Furthermore, multiplying elements of $\hat{G}$ by another element permutes the elements. Thus,

$$\sum_{\chi \in \hat{G}} \chi(g) = \sum_{\chi \in \hat{G}} \psi(g)\chi(g) = \psi(g) \sum_{\chi \in \hat{G}} \chi(g).$$

Since $\psi(g) \neq 1$, $\sum_{\chi \in \hat{G}} \chi(g) = 0$. □

Using lemma 0.1, we can obtain the far more general result

$$\sum_{\chi \in \hat{G}} \chi(g_1)\bar{\chi}(g_2) = \begin{cases} |G| & \text{if } g_1 = g_2, \\ 0 & \text{if } g_1 \neq g_2 \end{cases},$$

$$\sum_{g \in G} \chi_1(g)\bar{\chi}_2(g) = \begin{cases} |G| & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}.$$

The first result is obtained by letting $g = g_1 g_2^{-1}$ in the second part of lemma 0.1 and recognizing that $\chi\left(g^{-1}\right) = \bar{\chi}(g)$. The second result is obtained by letting $\chi = \chi_1 \bar{\chi}_2$ in the first part of lemma 0.1. If we now let $G = (\mathbb{Z}/m\mathbb{Z})^\times$, e get the following theorem for Dirichlet characters:

Theorem 1. *Let χ_1 and χ_2 be Dirichlet characters of $(\mathbb{Z}/m\mathbb{Z})^\times$, then we have that*

$$\sum_{\substack{1 \leq n \leq m-1 \\ \gcd(m,n)=1}} \chi_1(n)\bar{\chi}_2(n) = \begin{cases} \varphi(m) & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}.$$

Theorem 2. *Let χ denote a Dirichlet character for $(\mathbb{Z}/m\mathbb{Z})^\times$. Then for any $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$, we have*

$$\sum_{\chi} \chi(a)\bar{\chi}(b) = \begin{cases} \varphi(m) & \text{if } a \equiv b \pmod{m}, \\ 0 & \text{if } a \not\equiv b \pmod{m} \end{cases}.$$

Lastly, we will talk about how we can extend the domain of a Dirichlet character χ to all of $\mathbb{N}$. After all, we will eventually include our characters in a sum that ranges over all positive integers. In section 2, we saw that our definition of χ was not only defined over all natural numbers, but it was completely multiplicative. The most convenient way to extend an arbitrary Dirichlet character to $\mathbb{N}$ so that it still has these properties is to do the following:

Take a Dirichlet character χ for $(\mathbb{Z}/m\mathbb{Z})^\times$. Then for any $a \in \mathbb{N}$ and $b \in (\mathbb{Z}/m\mathbb{Z})^\times$

- Let $\chi(a) = \chi(b)$ if $a \equiv b \pmod{m}$ and $\gcd(a, m) = 1$
- Let $\chi(a) = 0$ if $\gcd(a, m) \neq 1$.

Notice that using our extended definition of Dirichlet characters, the sum in theorem 1 can just be taken over all integers from 0 to $m-1$ (i.e., all representatives modulo m) and the gcd restriction can be removed since χ will be zero for integers that aren't coprime to m anyway.

4 Dirichlet L-functions and Properties of Dirichlet Series

Given a complex-valued sequence $a_n : \mathbb{N} \to \mathbb{C}$, the Dirichlet series of a_n is defined to be the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s is a complex number such that the series converges. For the rest of this paper, whenever we see s, we will assume it is complex and denote its real part as σ and imaginary part as t. Much like how power series converge on a circle in the complex plane, Dirichlet series converge on half plane given by $\Re(s) > \sigma_0$. Dirichlet series converge absolutely on this half plane, and at the boundary $\Re(s) = \sigma_0$ they need not converge. However, if we know that a Dirichlet series converges at $s = \sigma_0$, then we know it must converge for $\Re(s) > \sigma_0$.

A Dirichlet L-function is a special case of a Dirichlet series, where our sequence a_n is a Dirichlet character (or rather the extended Dirichlet character over $\mathbb{N}$). As mentioned in section 1, we denote a Dirichlet L-function for a specific Dirichlet character χ as $L(\chi, s)$:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If a sequence a_n is multiplicative (i.e., $a_{mn} = a_m a_n$ whenever $\gcd(m, n) = 1$), then the Dirichlet series of a_n has an Euler product of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + ...\right),$$

which converges absolutely wherever the Dirichlet series of a_n converges absolutely. If a_n is completely multiplicative (i.e., $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$), then the Euler product takes the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - \frac{a_p}{p^s}}.$$

Since every Dirichlet character is completely multiplicative, its L-function permits an Euler product of the form

$$L(\chi, s) = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

The Riemann zeta function $\zeta(s)$ is the special case where we consider the (extended) trivial Dirichlet character for $(\mathbb{Z}/m\mathbb{Z})^\times$ when $m = 1$. In this case, the Dirichlet character is identically 1 for all positive integers, giving

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

for $\Re(s) > 1$. The zeta function clearly diverges for $s = 1$, but we characterize the function better, leading to the following theorem:

Theorem 3. *The Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1 and has an analytic continuation for $\sigma > 0$.*

Proof. We start with the following integral representation of the zeta function true for $\sigma > 1$:

$$\zeta(s) = \int_1^{\infty} \frac{s\lfloor x \rfloor}{x^{s+1}} dx,$$

Where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. If we let $\{x\}$ be the fractional part of x, so that $\lfloor x \rfloor = x - \{x\}$, then we have

$$\zeta(s) = \int_1^{\infty} s \left(\frac{1}{x^s} - \frac{\{x\}}{x^{s+1}} \right) dx = s \int_1^{\infty} \frac{1}{x^s} dx - \int_1^{\infty} \frac{s\,\{x\}}{x^{s+1}} dx.$$

If $\sigma > 1$, then $s \int_1^{\infty} \frac{1}{x^s} dx = \frac{s}{s-1} = \frac{1}{s-1} + 1$. Thus, we have that

$$\zeta(s) - \frac{1}{s-1} = 1 - \int_1^{\infty} \frac{s\,\{x\}}{x^{s+1}} dx.$$

The integral on the right is absolutely convergent for $\sigma > 0$. Moreover, the integral is analytic for $\sigma > 0$, thus $\zeta(s) - \frac{1}{s-1}$ extends to an analytic function for $\sigma > 0$. Since $\zeta(s) - \frac{1}{s-1}$ is analytic at $s = 1$, we have that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. □

In his paper on the prime-counting function, Riemann proved much more. He showed that $\zeta(s) - \frac{1}{s-1}$ can be analytically continued to the entire complex plane, showing that $\zeta(s)$ has only a single pole at $s = 1$. He even gave a functional equation that relates $\zeta(s)$ to $\zeta(1-s)$.

The product of two Dirichlet series is also a Dirichlet series and the new series can be expressed through Dirichlet convolution:

$$\left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \left(\sum_{k=1}^{\infty} \frac{b_n}{n^s} \right) = \sum_{m=1}^{\infty} \frac{c_m}{m^s},$$

where $c_n = a_n * b_n = \sum_{d|n} a_n b_{\frac{n}{d}}$.
The operation $a_n * b_n$ on two sequences a_n and b_n is called the Dirichlet convolution of a_n and b_n, and it is defined exactly as above. The convolution is a

linear, commutative, binary operation.

The complex conjugate of an L-function also takes a simple form:

$$\overline{L(\chi, s)} = \sum_{n=1}^{\infty} \overline{\left(\frac{X(n)}{n^s}\right)} = \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^{\bar{s}}} = L(\bar{\chi}, \bar{s})$$

The following two lemmas will be needed for the proof of Dirichlet's theorem:

Lemma 3.1. *If χ is a nontrivial Dirichlet character mod m, then $L(\chi, 1) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges for $\Re(s) > 0$*

Proof. If $\Re(s) > 1$, then $L(\chi, 1)$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

Let χ be a nontrivial Dirichlet character mod q and s be a real number such that $0 < s \leq 1$. Then, $\chi(n) = \chi(n + q)$ since $n \equiv n + q \pmod{q}$, so χ is q-periodic. Also, let $S(N) = \sum_{n=1}^{N} \chi(n)$ be the partial sums for χ. Then, we can write $N = kq + r$ for $0 \leq r < q$. Each integer n such that $1 \leq n \leq N$ can be written in the form $aq + b$. If $0 \leq a \leq k - 1$, then we can restrict $0 \leq b \leq q - 1$ for this representation to be unique. Since $N = kq + r$ and $n \leq N$, we must restrict $0 \leq b \leq r$ if $n = kq + r$ (i.e. $a = k$) and this representation will be unique. Thus,

$$S(N) = \sum_{a=0}^{k-1} \sum_{b=0}^{q-1} \chi(aq + b) + \sum_{b=0}^{r} \chi(kq + b).$$

Using the periodicity of χ, we have

$$S(N) = \sum_{a=0}^{k-1} \sum_{b=0}^{q-1} \chi(b) + \sum_{b=0}^{r} \chi(b).$$

Since χ is nontrivial, $\sum_{b=0}^{q-1} \chi(b) = 0$ by lemma 0.1. Thus, $S(N) = \sum_{b=0}^{r} \chi(b)$. Now, we can directly bound the sum:

$$|S(N)| \leq \sum_{b=0}^{r} |\chi(b)| \leq \sum_{b=0}^{r} 1 = r + 1 \leq q.$$

Thus, the partial sums of χ are bounded. Moreover, $\frac{1}{n^s}$ is a monotonic sequence that converges to 0, so $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges for real all real s such that $0 < s \leq 1$ by Dirichlet's test. Since this is a Dirichlet series, however, we have that it must converge for $\Re(s) > 0$ □

Lemma 3.2. *The exponential of an absolutely convergent Dirichlet series is an absolutely convergent Dirichlet series. Moreover, if $b_n \geq 0$ for all $n \in \mathbb{N}$ and $\exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n^s}\right) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, then $0 \leq b_n \leq a_n$.*

We will not prove lemma 3.2 and take that as given. We now have all the tools to prove Dirichlet's theorem.

5 The Non-Vanishing of $L(\chi, 1)$

In section 2, a pivotal step in proving Dirichlet's theorem for the special case of primes that are 1 and 3 mod 4 is the fact that $\log L(\chi, 1)$ was bounded. It was even more fortunate that we knew the exact value of $L(\chi, 1)$, but this won't necessarily be the case for a general nontrivial χ. Using lemma 3.1, we know that $L(\chi, 1)$ is finite for a nontrivial Dirichlet character χ. Thus, $\log L(\chi, 1)$ is bounded as long as $L(\chi, 1) \neq 0$. This is exactly what we will prove in this section, and we will see how Dirichlet's theorem falls cleanly from this result.

Before we proceed with the proof that $L(\chi, 1) \neq 0$ for any nontrivial χ, we will need the following theorem due to Landau:

Theorem 4 (Landau). *If $f(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ converges on $\sigma > \sigma_0$ with $a_n \geq 0$ for every n and f has an analytic continuation to the larger half-plane $\sigma > \sigma_1$, where $\sigma_1 < \sigma_0$, then $F(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ for $\sigma > \sigma_1$.*

With this, we can now prove the non-vanishing of $L(\chi, 1)$:

Suppose for contradiction that $L(\chi, 1) = 0$. For $\Re(s) > 0$ define

$$F(s) = \zeta(s)^2 L(\chi, s) L(\bar{\chi}, s).$$

This is analytic in s perhaps at $s = 1$. Since $\overline{L(\chi, s)} = L(\bar{\chi}, \bar{s})$, we have that $L(\chi, 1) = L(\bar{\chi}, 1) = 0$. Thus, $L(\chi, s)L(\bar{\chi}, s)$ has at least a double zero at $s = 1$, which would cancel with the double pole at $s = 1$ due to $\zeta(s)^2$. Thus, $F(s)$ is analytic for $\Re(s) > 0$. Now, we can show that the Dirichlet series of coefficients of $F(s)$ is nonnegative via its Euler product:

$$F(s) = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)^2} \frac{1}{1 - \frac{\chi(p)}{p^s}} \frac{1}{1 - \frac{\bar{\chi}(p)}{p^s}}$$

for $\Re(s) > 1$. We can then write each factor of the Euler product as an exponential of its logarithm:

$$\frac{1}{\left(1 - \frac{1}{p^s}\right)^2} \frac{1}{1 - \frac{\chi(p)}{p^s}} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \exp\left(\sum_{k=1}^{\infty} \frac{2 + \chi(p)^k + \bar{\chi}(p)^k}{kp^{ks}}\right).$$

Notice that $2 + \chi(p)^k + \bar{\chi}(p)^k$ is nonnegative. This is because either $\chi(p) = \bar{\chi}(p) = 0$ or $|\chi(p)| = |\bar{\chi}(p)| = 1$. If $\chi(p) = \bar{\chi}(p) = 0$, then $2 + \chi(p)^k + \bar{\chi}(p)^k = 0$. Otherwise, we can write $\chi(p) = e^{i\theta}$ and we have $2 + \chi(p) + \bar{\chi}(p) = 2(1 + \cos\theta) \geq 0$. Multiplying over all p gives

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{2 + X(p)^k + X(p)^k}{kp^{ks}}\right) = \exp\left(\sum_p \sum_{k=1}^{\infty} \frac{2 + X(p)^k + X(p)^k}{kp^{ks}}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n^s}\right),$$

where $\Re(s) > 1$ and $b_n \geq 0$. Since the exponential of a Dirichlet series with positive coefficients is also Dirichlet series with positive coefficients, we have

$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $a_n \geq 0$.

Now let $z = 1/p^s$ and $c = \chi(p)$. Then,

$$\frac{1}{\left(1 - \frac{1}{p^s}\right)^2} \frac{1}{1 - \frac{\chi(p)}{p^s}} \frac{1}{1 - \frac{\bar{\chi}(p)}{p^s}} = \frac{1}{(1-z)^2} \frac{1}{(1-cz)} \frac{1}{(1-\bar{c}z)}.$$

If we expand the above as a power series in z, we get

$$\frac{1}{(1-z)^2} \frac{1}{(1-cz)} \frac{1}{(1-\bar{c}z)} = 1 + (2 + c + \bar{c})\, z + \left(c^2 + c\bar{c} + \bar{c}^2 + 2c + 2\bar{c} + 3\right) z^2 + \ldots$$

We have already shown that the coefficient of z is nonnegative. The coefficient of z^k in the expansion above is the coefficient of $\frac{1}{p^{ks}}$ in $F(s)$, which is a_{p^k}. Moreover, the coefficient of z^2 is a_{p^2} and it is

$$\left(c^2 + c\bar{c} + \bar{c}^2 + 2c + 2\bar{c} + 3\right) = (c + c' + 1)^2 + 2 - |c|^2 \geq 2 - |c|^2 \geq 1.$$

Thus, $a_{p^2} \geq 1$ for every prime p. Since $F(s)$ is analytic for $\Re(s) > 0$, $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\Re(s) > 0$. Since $a_n \geq 0$ for all n,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \geq \sum_{p}^{\infty} \frac{a_{p^2}}{p^{2s}} \geq \sum_{p}^{\infty} \frac{1}{p^{2s}}.$$

However, we know $F(s)$ is analytic at $s = \frac{1}{2}$, but $\sum_p \frac{1}{p}$ diverges, as we saw in section 1. This is a contradiction, so $L(\chi, 1)$ is nonzero.

6 Proof of Dirichlet's Theorem

We now have all the tools to prove Dirichlet's theorem. The strategy will be the same as outlined in section 2. We will use the Dirichlet characters and their orthogonality relations to filter out the relevant primes. Then, we will show that the sums of the reciprocals of such primes diverges by showing that it equals sum of bounded terms and a single divergent term using the fact non-vanishing of $L(\chi, 1)$ for nontrivial χ.

Theorem 5 (Dirichlet's Theorem on Primes in Arithmetic Progression)**.** *If* m n *are two positive integers such that* $\gcd(m, n) = 1$*, then there are infinitely many primes* p *such that* $p \equiv n \pmod{m}$.

Proof. Let χ be a Dirichlet character modulo m. Then, we have that

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p} \frac{1}{1 - \frac{\chi(p)}{p^s}},$$

which converges absolutely for $\sigma > 1$. If $\chi = \chi_0$ is trivial, then

$$L(\chi_0, s) = \sum_{\substack{n \geq 1 \\ \gcd(m,n)=1}} \frac{1}{n^s},$$

which diverges at $s = 1$ by the limit comparison test with the harmonic series.

If χ is nontrivial, then $L(\chi, s)$ converges for $\sigma > 0$ and is nonzero at $s = 1$ Thus, $\log L(\chi, s)$ is bounded for $s \to 1^+$ if χ is nontrivial and $\log L(\chi, s)$ grows unboundedly for $s \to 1^+$. Taking logarithm of $L(\chi, s)$, we get

$$\log L(\chi, s) = \sum_p \sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} = \sum_p \frac{\chi(p)}{p^s} + \sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^{ks}}.$$

We can directly bound the last sum on the right:

$$\left| \sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right| \leq \sum_p \sum_{k=2}^{\infty} \frac{|\chi(p)|^k}{kp^{k\sigma}} \leq \sum_p \sum_{k=2}^{\infty} \frac{1}{p^{k\sigma}} = \sum_p \frac{1}{p^{\sigma}(p^{\sigma} - 1)}.$$

Instead of summing over primes we can sum over all natural numbers greater than 1 and this will give us a larger sum:

$$\sum_p \frac{1}{p^{\sigma}(p^{\sigma} - 1)} \leq \sum_{n=2}^{\infty} \frac{1}{n^{\sigma}(n^{\sigma} - 1)} \leq \sum_{n=2}^{\infty} \frac{1}{(n^{\sigma} - 1)^2}.$$

It can be shown, using limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}}$ for example, that $\sum_{n=2}^{\infty} \frac{1}{(n^{\sigma}-1)^2}$ converges for $\sigma > \frac{1}{2}$. Therefore, $\sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^{ks}}$ is bounded for $\sigma > \frac{1}{2}$ and we will call this sum $g(\chi, s)$. Therefore, we have

$$\log L(\chi, s) = \sum_p \frac{\chi(p)}{p^s} + g(\chi, s),$$

where g is bounded for $s \to 1$. We can then multiply both sides by the conjugate character evaluated at n, where $\gcd(n, m) = 1$, and sum over all Dirichlet characters mod m:

$$\sum_{\chi} \bar{\chi}(n) \log L(\chi, s) = \sum_{\chi} \sum_p \frac{\bar{\chi}(n)\chi(p)}{p^s} + \sum_{\chi} \bar{\chi}(n)\, g(\chi, s).$$

Using the orthogonality relation from theorem 2, we see that the first sum on the right side of the equation simplifies to a sum over those primes $p \equiv n \pmod{m}$. Thus,

$$\phi(m) \sum_{p \equiv n \pmod{m}} \frac{1}{p^s} = \sum_{\chi} \bar{\chi}(n) \log L(\chi, s) - \sum_{\chi} \bar{\chi}(n)\, g(\chi, s)$$

$$= \chi_0(n) \log L(\chi_0, s) + \sum_{\chi \neq \chi_0} \bar{\chi}(n) \log L(\chi, s) - \sum_{\chi} \bar{\chi}(n) g(\chi, s).$$

Since $g(\chi, s)$ is bounded for all $\sigma > \frac{1}{2}$ and $\log L(\chi, s)$ is bounded for all $\sigma > 0$ whenever $\chi \neq \chi_0$, the last 2 terms are bounded for $s \to 1$. However, since $L(\chi_0, 1)$ diverges, the first term is not bounded and we have

$$\sum_{p \equiv n \pmod m} \frac{1}{p} \text{ is divergent.}$$

It might seem like we didn't require n to be coprime to m, but we actually did when we multiplied by $\bar{\chi}(n)$. If $\gcd(n, m) \neq 1$, then $\bar{\chi}(n) = 0$ and we would not be able to get information on any of the quantities in question. Mainly, $\chi_0(n) \log L(\chi_0, s) = 0$ for all $\sigma > 1$, and we wouldn't have been able to conclude anything about the divergence of this term. The only way a Dirichlet character mod m is nonzero is if its input is coprime to m. □